Asset Pricing

Chapter VIII. Arrow-Debreu Pricing

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Setting: An Arrow Debreu Economy

General Equilibrium

- Supply = Demand
- With production (if desired)
- Static but Multi-period
- No restrictions on preferences

The most general of the theories we shall consider

- 1. Two dates: 0, 1 . This set-up, however, is fully generalizable to multiple periods.
- 2. N possible states of nature at date 1, which we index by $\theta = 1, 2, ..., N$ with probabilities π_θ
- 3. One perishable (non storable) consumption good
- 4. K agents, indexed k = 1,...,K, with preferences:

$$U_0^k\left(c_0^k\right) + \delta^k \sum_{\theta=1}^{N} \pi_\theta U^k\left(c_\theta^k\right);$$

- 5. In addition, agent k's endowment is described by the vector $\{e_0^k, \left(e_\theta^k\right)_{\theta=1,2,...,N}\}$ Possibly:

$$u^k(c_0^k, c_{\theta_1}^k, c_{\theta_2}^k, ..., c_{\theta_N}^k).$$

Traded Securities

- Exclusively Arrow-Debreu securities (contingent claims): Security θ priced q_θ promises delivery of one unit of commodity tomorrow if state θ is realized and nothing otherwise
- How to secure one unit of consumption tomorrow?

$$\max_{(c_0^k, c_1^k, \ldots, c_N^k)} U_0^k(c_0^k) + \delta^k \sum_{\theta=1}^{N} \pi_\theta U^k(c_\theta^k)$$

s.t.

$$c_0^k + \sum_{\theta=1}^{N} q_\theta c_\theta^k \leq e_0^k + \sum_{\theta=1}^{N} q_\theta e_\theta^k \qquad \text{(P)}$$

$$c_0^k, c_1^k, \ldots, c_N^k \geq 0$$

Equilibrium is a set of prices $(q_1, \ldots q_N)$ such that:

- 1. at those prices $(c_0^k, \ldots, c_N^k)$ solve problem (P), for all k, and
- 2. $\sum_{k=1}^{K} c_0^k = \sum_{k=1}^{K} e_0^k$, $\sum_{k=1}^{K} C_\theta^k = \sum_{k=1}^{K} e_\theta^k$, for every θ

Table 8.1. Endowments and Preferences in Our Reference Example

Agents	Endowments			Preferences
	$t=0$	$t=1$		
		θ_1	θ_2	
Agent 1	10	1	2	$\frac{1}{2}c_0^1 + 0.9\left(\frac{1}{3}\ln\left(c_1^1\right) + \frac{2}{3}\ln\left(c_2^1\right)\right)$
Agent 2	5	4	6	$\frac{1}{2}c_0^2 + 0.9\left(\frac{1}{3}\ln\left(c_1^2\right) + \frac{2}{3}\ln\left(c_2^2\right)\right)$

The respective agent problems are:

Agent 1:

$$\max \frac{1}{2}\left(10+1q_1+2q_2-c_1^1q_1-c_2^1q_2\right)+0.9\left(\frac{1}{3}\ln\left(c_1^1\right)+\frac{2}{3}\ln\left(c_2^1\right)\right)$$
$$\text{S.t.}\quad c_1^1q_1+c_2^1q_2\leq 10+q_1+2q_2,\ \text{and } c_1^1,c_2^1\geq 0$$

Agent 2:

$$\max \frac{1}{2}\left(5+4q_1+6q_2-c_1^2q_1-c_2^2q_2\right)+0.9\left(\frac{1}{3}\ln\left(c_1^2\right)+\frac{2}{3}\ln\left(c_2^2\right)\right)$$
$$\text{S.t.}\quad c_1^2q_1+c_2^2q_2\leq 5+4q_1+6q_2 \text{ and } c_1^2,c_2^2\geq 0$$

Note: q_1,q_2 solve «key finance problem»!

$$\text{Agent 1}:\begin{cases} c_1^1:\ \frac{q_1}{2}=0.9\left(\Pi_\theta\right)\frac{1}{c_1^1} \\ c_2^1:\ \frac{q_2}{2}=0.9\left(\frac{2}{3}\right)\frac{1}{c_2^1}\end{cases}\qquad \text{Agent 2}:\begin{cases} c_1^2:\ \frac{q_1}{2}=0.9\left(\frac{1}{3}\right)\frac{1}{c_1^2} \\ c_2^2:\ \frac{q_2}{2}=0.9\left(\frac{2}{3}\right)\frac{1}{c_2^2}\end{cases}\qquad c_2^1+c_2^2=8$$

$$q_\theta = \frac{\delta \pi_\theta \frac{\partial U^k}{\partial c_\theta^k}}{\frac{\partial U_0^k}{\partial c_0^k}}, \; k, \theta = 1, 2 \tag{1}$$

$$\frac{\text{price of gold if state } \theta \text{ is realized}}{\text{price of gold today}} = \frac{MU_\theta^k}{MU_0^k},$$

$$c_1^1 = c_1^2 = 2.5$$
$$c_2^1 = c_2^2 = 4$$

$$q_1 = 2\,(0.9)\left(\tfrac{1}{3}\right)\left(\frac{1}{c_1^1}\right) = 2\,(0.9)\left(\tfrac{1}{3}\right)\left(\tfrac{1}{2.5}\right) = (0.9)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{5}\right) = 0.24$$
$$q_2 = 2\,(0.9)\left(\tfrac{2}{3}\right)\left(\frac{1}{c_2^1}\right) = 2\,(0.9)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{4}\right) = (0.9)\left(\tfrac{2}{3}\right)\left(\tfrac{4}{8}\right) = 0.3$$

Table 8.2: Post-Trade Equilibrium Consumptions

	t = 0	t = 1	
		θ_1	θ_2
Agent 1:	9.04	2.5	4
Agent 2:	5.96	2.5	4
Total	15.00	5.0	8

It is Pareto optimal? H1 and H2 are satisfied!

Checking Pareto Optimality

$$\max_{\{c_0^1, c_1^1, c_2^1\}} u^1(c_0^1, c_1^1, c_2^1) + \lambda u^2(c_0^2, c_1^2, c_2^2)$$

s.t.

$$c_0^1 + c_0^2 = 15; c_1^1 + c_1^2 = 5; c_2^1 + c_2^2 = 8,$$

$$c_0^1, c_1^1, c_2^1, c_0^2, c_1^2, c_2^2 \geq 0$$

$$\frac{u_0^1}{u_0^2} = \frac{u_1^1}{u_1^2} = \frac{u_2^1}{u_2^2} = \lambda \tag{2}$$

$$\frac{1/2}{1/2} = \frac{(0.9)\frac{1}{3}\frac{1}{c_1^1}}{(0.9)\frac{1}{3}\frac{1}{c_1^2}} = \frac{(0.9)\frac{2}{3}\frac{1}{c_2^1}}{(0.9)\frac{2}{3}\frac{1}{c_2^2}}$$

Box 8-1 Interior vs. Corner Solution

$$q_0 \frac{\partial U_0^k}{\partial c_0^k} \leq \delta \pi_\theta \frac{\partial U^k}{\partial c_\theta^k}, \text{ if } c_0^k > 0, \text{ and } k, \theta = 1, 2 \tag{3}$$

If one security (q_1) only (incomplete markets):

Table 8.3 : The Post-Trade Allocation

	t = 0	t = 1	
		θ_1	θ_2
Agent 1:	9.64	2.5	2
Agent 2:	5.36	2.5	6
Total	15.00	5.0	8

- Agent 1: 5.51 instead of 5.62
- Agent 2: 4.03 instead of 4.09

Table 8.4: The New Endowment Matrix

	t = 0	t = 1	
		θ_1	θ_2
Agent 1	4	1	5
Agent 2	4	5	1

Table 8.5: Agents' Utility In The Absence of Trade

	State-Contingent Utility		Expected Utility in Period 1
	θ_1	θ_2	
Agent 1	ln(1) = 0	ln(5) = 1.609	1/2 ln(1) + 1/2 ln(5) = 8047
Agent 2	ln(5) = 1.609	ln(1) = 0	1/2 ln(1) + 1/2 ln(5) = 8047

Table 8.6: The Desirable Trades And Post-Trade Consumptions

Date 1	Endowments Pre-Trade		Consumption Post-Trade	
	θ_1	θ_2	θ_1	θ_2
Agent 1	1	5 [$\Downarrow$2]	3	3
Agent 2	5 [$\Uparrow$2]	1	3	3

- Post-trade allocation is Pareto Optimal
- After trade, EU in period 1 is approx. 1.1 for both agents.

Table 8.7 : Another Set of Initial Allocations

	t = 0	t = 1	
		θ_1	θ_2
Agent 1	4	1	3
Agent 2	4	5	3

Table 8.8 : Plausible Trades And Post-Trade Consumptions

Date 1	Endowments Pre-trade		Consumption Post-trade	
	θ_1	θ_2	θ_1	θ_2
Agent 1	1	3 [$\Downarrow$1]	2	2
Agent 2	5 [$\Uparrow$1]	3	4	4

- Post-trade allocation is PO
- Features perfect risk sharing or full mutual insurance (no aggregate risk)

$$\frac{u_0^1}{u_0^2} = \frac{u_1^1}{u_1^2} = \frac{u_2^1}{u_2^2} = \lambda \tag{4}$$

$$\frac{u_1^1}{u_1^2} = \frac{u_2^1}{u_2^2} = \frac{u_1^1}{u_2^1} = \frac{u_1^2}{u_2^2} \tag{5}$$

1. If one of the two agents is fully insured, the other must be as well.
2. More generally, if the MRS are to differ from q, given that they must be equal between them, the low consumption - high MU state must be the same for both agents and similarly for the high consumption - low MU state.
3. If there is aggregate risk, however, the above reasoning also implies that, at a Pareto optimum, it is shared "proportionately" among agents with same risk tolerance.
4. Finally, if agents are differentially risk averse, in a Pareto optimal allocation the less risk averse will typically provide some insurance services to the more risk averse.
5. More generally, optimal risk sharing dictates that the agent most tolerant of risk bears a disproportionate share of it.

Implementing Pareto Optimal Allocations: On the Possibility of Market failure

(i) Agent 1 solves: $\max\ (4 - q_Q z_Q^1) + [1/2 ln(1 + z_Q^1) + 1/2 ln(5)]$
s.t. $q_Q z_Q^1 \leq 4$

(ii) Agent 2 solves: $\max\ (4 - q_Q z_Q^2) + [1/2 ln(5 + z_Q^2) + 1/2 ln(1)]$
s.t. $q_Q z_Q^2 \leq 4$

Assuming an interior solution, the FOCs are

(i)' : $-q_Q + \frac{1}{2}\left(\frac{1}{1+z_Q^1}\right) = 0;$

$(ii)'$: $-q_Q + \frac{1}{2}\left(\frac{1}{5+z_Q^2}\right) = 0 \Rightarrow \frac{1}{1+z_Q^1} = \frac{1}{5+z_Q^2};$

Table 8.9: Market Allocation When Both Securities Are Traded

	t = 0	t = 1	
		θ_1	θ_2
Agent 1	4	3	3
Agent 2	4	3	3

Table 8.10: The Net Gains From Trade
Expected Utility Levels and Net Trading Gains
(Gain to issuer in bold)

	No Trade	Trade Only Q		Trade Both $Q$and R	
	EU	EU	ΔEU$^{(i)}$	EU	ΔEU$^{(ii)}$
Agent 1	4.8047	5.0206	0.2159	5.0986	**0.0726**
Agent 2	4.8047	4.883	**0.0783**	5.0986	0.2156
Total			0.2942		0.2882

- The «father» of all asset pricing relationships
- A reference
- Conceptual import
- Hard to identify pure state of nature
- Static